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# Bound states in tubular quantum waveguides with torsion

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**Abstract.** We consider a quantum particle constrained to move within a tube of central radius $d$ embedded in 3-space, subject to Dirichlet boundary conditions. Taking the central axis of the tube as a reference curve and setting up a locally cylindrical polar coordinate system around this curve, we derive an exact expression for the effective potential introduced by the imposition of curvature and torsion. We then employ a minimax method to ascertain a sufficient requirement for the curvature and torsion to guarantee the existence of a bound state.

## 1. Introduction

Recent advances in microelectronics and semiconductor microengineering have enabled the fabrication of reduced-dimensional quantum systems such as quantum wells and quantum wires [1, 2]. Two-dimensional layered quantum systems such as quantum wells are already being used in practical applications, for example to construct lasers in compact disc players and other optoelectronic applications.

One-dimensional systems such as quantum wires, in which a quantum particle is constrained to move along a one-dimensional trajectory (where the transverse modes are effectively limited to the ground state by energy considerations) are somewhat more difficult to fabricate. The advantage of one-dimensional systems is that small-angle scattering should be greatly diminished in comparison with two-dimensional systems, which would enable bandgap-engineered structures such as semiconductor lasers to be fabricated with substantially improved performance.

So far, experimental one-dimensional systems have been manufactured typically by processing two-dimensional layered quantum systems, such as by evaporative deposition of patterned metal gates onto a surface to cause electron depletion in certain regions through application of a negative electrical potential to the gate electrodes. The split-gate device is an example of how this technique can be successful.

However, to realize properly the advantages of one-dimensional nanostructure engineering, techniques which allow the fabrication of quantum wires embedded with torsion in three dimensions will allow processing units with much improved connectivity to be developed. Freeing quantum wires from the confines of a planar surface will allow nanostructure devices with new forms to be created. Exner and Šeba [3] suggest that quantum wires could theoretically be etched in a helical fashion around the surface of a thin cylindrical rod, thus producing one such new device for which we might coin the phrase *quantum solenoid*.

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Several authors [4–6] have modelled the dynamics of a non-relativistic quantum particle constrained to move along a curve. However, due to the engineering problems in constructing quantum wires, it is somewhat ambitious at this stage to assume that such waveguides can be constructed with sufficiently small thickness that the transverse dimension can be neglected entirely. Modelling of a non-relativistic quantum particle confined to move along a thin tubular neighbourhood of a curve embedded in 3-space has been previously considered by several authors [7–12]

With these motivations, we consider here the quantum dynamics of a particle constrained to move within a tube of mesoscopic, but not negligible, thickness $d$ embedded with torsion in 3-space.

Further, we establish some results regarding the effect of torsion on the existence of bound states for such a system by use of the minimax technique.

## 2. The coordinate system

Let a reference curve $\mathcal{C}$ be described in 3-space by a smooth vector-valued function $\boldsymbol{r}$ of arc-length $q_1$:

$$\mathcal{C} = \{\boldsymbol{r}(q_1) : q_1 \in \mathbb{R}\}. \tag{1}$$

Along this reference curve $\mathcal{C}$, the unit tangent vector is given by the first derivative of $\boldsymbol{r}$

$$\boldsymbol{t}(q_1) = \frac{\boldsymbol{r}'(q_1)}{\|\boldsymbol{r}'(q_1)\|} = \boldsymbol{r}'(q_1) \tag{2}$$

where we note that this choice of $q_1$ as arc length requires that $\|\boldsymbol{r}'(q_1)\| = 1$. We will assume that $\mathcal{C}$ is suitably smooth so as to ensure that $\boldsymbol{r}(q_1)$ is twice differentiable everywhere. Then we obtain the curvature $\kappa(q_1)$ of $\mathcal{C}$ from the second derivative of $\boldsymbol{r}$,

$$\kappa(q_1) = \|\boldsymbol{r}''(q_1)\|. \tag{3}$$

We impose the condition $\kappa(q_1)d \ll 1$ in order to guarantee smoothness of the boundary of the tube. We define $\kappa_+ = \sup_{q_1} \kappa(q_1)$, where $\kappa_+ d \ll 1$.

Unit normal and binormal vectors are then defined in the usual manner:

$$\boldsymbol{n}(q_1) = \frac{\boldsymbol{t}'(q_1)}{\kappa(q_1)} \tag{4}$$

$$\boldsymbol{b}(q_1) = \boldsymbol{t}(q_1) \wedge \boldsymbol{n}(q_1) \tag{5}$$

so that the set $\{\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}\}$ forms a right-handed orthonormal triad. The torsion $\tau(q_1)$ of $\mathcal{C}$ is then given by one of the Frenet–Serret formulae,

$$\boldsymbol{b}'(q_1) = -\tau(q_1) \qquad \boldsymbol{n}(q_1) \tag{6}$$

and characterizes the tendency for the curve to twist out of the osculating plane (the plane of $\boldsymbol{t}$ and $\boldsymbol{n}$). We define the total twisting $T(q_1)$ of $\mathcal{C}$ relative to a reference point $\boldsymbol{r}(q_0)$ along the curve by

$$T(q_1) = \int_{q_0}^{q_1} \tau(u)\,\mathrm{d}u. \tag{7}$$

For the purposes of the analytical results in section 4, we will impose certain decay assumptions on $\kappa(q_1)$ and $\tau(q_1)$. Since these results do not alter the validity of the effective potential term we derive in section 3, we postpone the introduction of these decay assumptions until they are required.

The position vector $\boldsymbol{R}$ of an arbitrary point in the vicinity of $\mathcal{C}$ can be expressed in terms of three coordinates $q_1, q_2, \theta$, where the first two of these coordinates have the dimension of length and the third is a dimensionless angular coordinate:

$$\boldsymbol{R}(q_1, q_2, \theta) = \boldsymbol{r}(q_1) + q_2 \cos(\theta - T(q_1))\boldsymbol{n}(q_1) + q_2 \sin(\theta - T(q_1))\boldsymbol{b}(q_1) \tag{8}$$

$$= \boldsymbol{r}(q_1) + q_2 C\boldsymbol{n}(q_1) + q_2 S\boldsymbol{b}(q_1) \tag{9}$$

using the shorthand $C = \cos(\theta - T(q_1))$, $S = \sin(\theta - T(q_1))$.

From (9) we have

$$\|d\boldsymbol{R}\|^2 = [1 - \kappa q_2 \cos(\theta - T(q_1))]^2 \, \mathrm{d}q_1^2 + \mathrm{d}q_2^2 + q_2^2 \, \mathrm{d}\theta^2 \tag{10}$$

so that $q_1$, $q_2$ and $\theta$ are a set of orthogonal curvilinear coordinates with scale factors $h_1 = 1 - \kappa q_2 \cos(\theta - T(q_1))$, $h_2 = 1$ and $h_\theta = q_2$. Note that we have a singularity in the coordinate system when $q_2 = 0$, i.e. for points directly on $\mathcal{C}$. We then construct the Laplace–Beltrami operator

$$\nabla^2 = \frac{1}{h_1 h_2 h_\theta} \left[ \frac{\partial}{\partial q_1} \left( \frac{h_2 h_\theta}{h_1} \frac{\partial}{\partial q_1} \right) + \frac{\partial}{\partial q_2} \left( \frac{h_1 h_\theta}{h_2} \frac{\partial}{\partial q_2} \right) + \frac{\partial}{\partial \theta} \left( \frac{h_1 h_2}{h_\theta} \frac{\partial}{\partial \theta} \right) \right] \tag{11}$$

which reduces here to

$$\nabla^2 = \frac{1}{h_1^2} \frac{\partial^2}{\partial q_1^2} - \frac{1}{h_1^3} \frac{\partial h_1}{\partial q_1} \frac{\partial}{\partial q_1} + \frac{\partial^2}{\partial q_2^2} + \left( \frac{1}{q_2} + \frac{1}{h_1} \frac{\partial h_1}{\partial q_2} \right) \frac{\partial}{\partial q_2} + \frac{1}{q_2^2} \frac{\partial^2}{\partial \theta^2} + \frac{1}{h_1 q_2^2} \frac{\partial h_1}{\partial \theta} \frac{\partial}{\partial \theta}. \tag{12}$$

Note that if $\kappa(q_1) = 0$, then $h_1$ reduces to unity and (12) reduces to the usual cylindrical Laplacian. Since (12) only involves the single scale factor $h_1$, we follow Exner [8] and use the notation $h = h_1$ for brevity.

## 3. The Hamiltonian and the effective potential

Our objective is to solve the Schrödinger equation for a free quantum particle of effective mass $m^*$, written in the form

$$H_\Omega \psi(q_1, q_2, \theta) \equiv \frac{-\hbar^2}{2m^*} \nabla^2 \psi(q_1, q_2, \theta) = E\psi(q_1, q_2, \theta) \tag{13}$$

for $q_1 \in \mathbb{R}$, $q_2 \in [0, d]$ and $\theta \in [0, 2\pi)$, where $H_\Omega$ is the Hamiltonian for the quantum waveguide subject to Dirichlet boundary conditions

$$\psi(q_1, d, \theta) = 0 \qquad \forall q_1 \in \mathbb{R},\ \theta \in [0, 2\pi). \tag{14}$$

We now make the substitution

$$\psi(q_1, q_2, \theta) = h^{-1/2} \chi(q_1, q_2, \theta) \tag{15}$$

in order to transform the sizes of the volume elements $\mathrm{d}V$ in this curvilinear coordinate system back into the usual volume element in straightened cylindrical polar coordinates. This unitary transformation straightens out the coordinate system in a way which allows us to decompose the kinetic energy component of the Hamiltonian into a longitudinal component and a transverse component. The price of this simplification of the kinetic part of the Hamiltonian is that in the transformed coordinate system, an effective potential energy term must be introduced.

Constructing the Hamiltonian $H_0$ for this system by transforming (13) using (12) and (15), we have

$$H_0 \chi(q_1, q_2, \theta) = E\chi(q_1, q_2, \theta) \tag{16}$$

where $H_0$ is given by

$$H_0\chi = \frac{-\hbar^2}{2m^*}\left[\frac{\partial}{\partial q_1}\left(\frac{1}{h^2}\frac{\partial\chi}{\partial q_1}\right) + \frac{\partial^2\chi}{\partial q_2^2} + \frac{1}{q_2}\frac{\partial\chi}{\partial q_2} + \frac{1}{q_2^2}\frac{\partial^2\chi}{\partial\theta^2}\right] + V_{\text{eff}}(q_1, q_2, \theta)\chi \tag{17}$$

subject to the imposition of an effective potential term in accordance with that of Exner [8]

$$V_{\text{eff}}(q_1, q_2, \theta) = \frac{-\hbar^2}{2m^*}\left(\frac{\kappa^2}{4h^2} - \frac{1}{2h^3}\frac{\partial^2 h}{\partial q_1^2} + \frac{5}{4h^4}\left(\frac{\partial h}{\partial q_1}\right)\right) \tag{18}$$

which we can express in terms of curvature and torsion as

$$V_{\text{eff}} = \frac{-\hbar^2}{2m^*}\left(\frac{\kappa^2}{4(1-\kappa q_2 C)^2} + \frac{q_2([\kappa'' - \kappa\tau^2]C + [\kappa\tau' + 2\kappa'\tau]S)}{2(1-\kappa q_2 C)^3} + \frac{5}{4}\frac{q_2^2(\kappa\tau S + \kappa' C)^2}{(1-\kappa q_2 C)^4}\right). \tag{19}$$

It is apparent from (19) that the particle energetically favours motion towards the edge of the tube, i.e. as $q_2 \to d$, because the contribution of the first and third terms in (19) towards the potential energy for the quantum particle are increasingly attractive for increasing values of $q_2$. This can be seen by taking Taylor series expansions about $q_2 = 0$ for the various terms. Hence, the thin and slowly twisting tube approximation [12] is difficult to justify if the waveguide itself is *not* thin or slowly twisting.

## 4. Existence of bound states

In order to establish criteria by which we can show when such a twisted quantum waveguide definitely has at least one bound state, we construct from $H_0$ new self-adjoint operators $H_+$ and $H_-$ with identical essential spectra $\sigma_{\text{ess}}(H_+) = \sigma_{\text{ess}}(H_0) = \sigma_{\text{ess}}(H_-) = [E_\infty, \infty)$, and for which $H_- \leqslant H_0 \leqslant H_+$ in the sense of quadratic forms:

$$\langle\psi|H_-\psi\rangle \leqslant \langle\psi|H_0\psi\rangle \qquad \forall\psi \in Q(H_-)\cap Q(H_0) \tag{20}$$

$$\langle\psi|H_0\psi\rangle \leqslant \langle\psi|H_+\psi\rangle \qquad \forall\psi \in Q(H_0)\cap Q(H_+) \tag{21}$$

where $Q(H)$ is the form domain of $H$. By the minimax principle [13], we then have

*Proposition 4.1.* If $H_+$ has an eigenvalue $E_0^+ < E_\infty$ then $H_0$ will necessarily have an eigenvalue $E_0$ for which $E_0 \leqslant E_0^+$. Furthermore, if $H_-$ has no eigenvalues below $E_\infty$, then $H_0$ will have no bound states.

From (17), $H_0$ can be decomposed into a sum of longitudinal and transverse kinetic energy operators, plus the effective potential

$$H_0 = H_{12} + H_2 + V_{\text{eff}} \tag{22}$$

where the longitudinal component of the Hamiltonian

$$H_{12} = \frac{-\hbar^2}{2m^*}\left[\frac{\partial}{\partial q_1}\left(\frac{1}{h^2}\frac{\partial}{\partial q_1}\right)\right] \tag{23}$$

depends on the curvature and torsion of $\mathcal{C}$ and upon the transverse coordinates $q_2$ and $\theta$ through the presence of $h$. However, the transverse component of the Hamiltonian

$$H_2 = \frac{-\hbar^2}{2m^*}\left[\frac{\partial^2}{\partial q_2^2} + \frac{1}{q_2}\frac{\partial}{\partial q_2} + \frac{1}{q_2^2}\frac{\partial^2}{\partial\theta^2}\right] \tag{24}$$

is the same differential operator as for a straight cylindrical quantum waveguide, and hence admits the same transverse mode eigenfunctions.

We construct $H_+$ and $H_-$ by defining suitable upper and lower bounds $V_+(q_1)$ and $V_-(q_1)$ for the effective potential,

$$V_-(q_1) \leqslant V_{\text{eff}}(q_1, q_2, \theta) \leqslant V_+(q_1) \qquad \forall q_1, q_2 \in \mathbb{R},\ \theta \in [0, 2\pi] \tag{25}$$

and suitable self-adjoint operators $H_1^{\pm}$ for which $H_1^- \leqslant H_{12} \leqslant H_1^+$ in the form sense on a common core. We then set

$$H_{\pm} = H_1^{\pm} + H_2 + V_{\pm}. \tag{26}$$

It remains only to specify $V_+(q_1)$, $V_-(q_1)$ and $H_1^{\pm}$.

*Proposition 4.2.* Let $\mu(q_1, q_2, \theta) = h^{-1} = [1 - \kappa(q_1)q_2\cos(\theta - T(q_1))]^{-1}$. Then there exist constants $\mu_- = (1 + \kappa_+ d)^{-1}$ and $\mu_+ = (1 - \kappa_+ d)^{-1}$, where $\kappa_+ = \sup_{q_1} \kappa(q_1)$, such that

$$\mu_-^n \leqslant \mu^n(q_1, q_2, \theta) \leqslant \mu_+^n \qquad \forall q_1 \in \mathbb{R},\ q_2 \in [0, d],\ \theta \in [0, 2\pi],\ n \geqslant 1. \tag{27}$$

*Theorem 4.3.* The operators $H_1^{\pm}$ defined by

$$H_1^{\pm} = \frac{-\hbar^2}{2m^*}\mu_{\pm}^2 \frac{\partial^2}{\partial q_1^2} \tag{28}$$

satisfy $H_1^- \leqslant H_{12} \leqslant H_1^+$, in the form sense on a common core.

*Proof.* We will show only that $\langle\psi|H_1^+\psi\rangle \geqslant \langle\psi|H_{12}\psi\rangle$, for it follows in the same manner that $\langle\psi|H_{12}\psi\rangle \geqslant \langle\psi|H_1^-\psi\rangle$.

Let $\psi \in Q(H_1^+) \cap Q(H_{12})$. Then

$$\begin{aligned}
\langle\psi|H_1^+\psi\rangle - \langle\psi|H_{12}\psi\rangle &= \frac{\hbar^2}{2m^*}\left[\left\langle\psi\left|\frac{\partial}{\partial q_1}\left(\mu^2\frac{\partial\psi}{\partial q_1}\right)\right.\right\rangle - \mu_+^2\left\langle\psi\left|\frac{\partial^2\psi}{\partial q_1^2}\right.\right\rangle\right] \\
&= \frac{\hbar^2}{2m^*}\int_0^{2\pi}\int_0^d\left[\int_{-\infty}^{\infty}\overline{\psi}\frac{\partial}{\partial q_1}\left(\mu^2\frac{\partial\psi}{\partial q_1}\right)\mathrm{d}q_1 - \mu_+^2\int_{-\infty}^{\infty}\overline{\psi}\frac{\partial^2\psi}{\partial q_1^2}\,\mathrm{d}q_1\right]\mathrm{d}q_2\,\mathrm{d}\theta \\
&= \frac{\hbar^2}{2m^*}\int_0^{2\pi}\int_0^d\left[\int_{-\infty}^{\infty}\overline{\psi}'(\mu_+^2 - \mu^2)\psi'\,\mathrm{d}q_1 - [\overline{\psi}(\mu_+^2 - \mu^2)\psi']_{-\infty}^{\infty}\right]\mathrm{d}q_2\,\mathrm{d}\theta \\
&= \frac{\hbar^2}{2m^*}\int_0^{2\pi}\int_0^d\int_{-\infty}^{\infty}\overline{\psi}'(\mu_+^2 - \mu^2)\psi'\,\mathrm{d}q_1\,\mathrm{d}q_2\,\mathrm{d}\theta \\
&= \langle\psi'|(\mu_+^2 - \mu^2)|\psi'\rangle \geqslant 0 \qquad \text{by proposition 4.2.}
\end{aligned}$$

□

*Theorem 4.4.* The following potentials $V_{\pm}(q_1)$ satisfy (25), where we use $\max\{a, b\} = \frac{1}{2}[a + b + |a - b|]$

$$V_+(q_1) = \frac{-\hbar^2}{2m^*}\left(\frac{\kappa^2}{4(1 + \kappa_+ d)^2} - \frac{d}{2(1 - \kappa_+ d)^3}\max\{|\kappa'' - \kappa\tau^2|, |\kappa\tau' + 2\kappa'\tau|\}\right) \tag{29}$$

$$\begin{aligned}
V_-(q_1) = \frac{-\hbar^2}{2m^*}\bigg(&\frac{\kappa^2}{4(1 - \kappa_+ d)^2} + \frac{d}{2(1 - \kappa_+ d)^3}\max\{|\kappa'' - \kappa\tau^2|, |\kappa\tau' + 2\kappa'\tau|\} \\
&+ \frac{5}{4}\frac{d^2}{(1 - \kappa_+ d)^4}\max\{(\kappa\tau)^2, (\kappa')^2\}\bigg).
\end{aligned} \tag{30}$$

*Proof.* This is a simple exercise in inequalities. We will demonstrate that $V_{\mathrm{eff}}(q_1, q_2, \theta) \geqslant V_-(q_1)$; the proof that $V_{\mathrm{eff}}(q_1, q_2, \theta) \leqslant V_+(q_1)$ proceeds in the same manner. Write (19) as

$$V_{\mathrm{eff}} = \frac{-\hbar^2}{2m^*}\left(\tfrac{1}{4}\kappa^2\mu^2 + \tfrac{1}{2}q_2\mu^3([\kappa'' - \kappa\tau^2]C + [\kappa\tau' + 2\kappa'\tau]S) + \tfrac{5}{4}q_2^2\mu^4(\kappa\tau S + \kappa' C)^2\right)$$

By proposition 4.2, we have

$$\begin{aligned} V_{\mathrm{eff}} &\geqslant \frac{-\hbar^2}{2m^*}\Bigg(\tfrac{1}{4}\kappa^2\mu_+^2 + \tfrac{1}{2}\mu_+^3 d \sup_{\theta\in[0,2\pi]}([\kappa'' - \kappa\tau^2]C + [\kappa\tau' + 2\kappa'\tau]S) \\ &\qquad + \tfrac{5}{4}\mu_+^4 d^2 \sup_{\theta\in[0,2\pi]}((\kappa\tau S + \kappa' C)^2)\Bigg) \\ &= \frac{-\hbar^2}{2m^*}\Bigg(\tfrac{1}{4}\kappa^2\mu_+^2 + \tfrac{1}{2}\mu_+^3 d \max\{|\kappa'' - \kappa\tau^2|, |\kappa\tau' + 2\kappa'\tau|\} \\ &\qquad + \tfrac{5}{4}\mu_+^4 d^2 \max\{(\kappa\tau)^2, (\kappa')^2\}\Bigg) \\ &= V_-(q_1). \end{aligned}$$

□

Having characterized the operators $H_1^{\pm}$ and the potentials $V_{\pm}$, we are now able to investigate the spectrum of $H_{\pm}$. It is here that we must impose decay assumptions upon $\kappa(q_1)$ and $\tau(q_1)$ which guarantee that $V_{\pm}(q_1)$ is locally square integrable, $\int_{-\infty}^{\infty} V_+(q_1)(1 + |q_1|)\,\mathrm{d}q_1 < \infty$ and that

$$\lim_{a\to\infty}\int_a^{a+1} |V_{\pm}(q_1)|^2\,\mathrm{d}q_1 = 0 \qquad \text{as } a \to \pm\infty. \tag{31}$$

With this, we have $\sigma_{\mathrm{ess}}(H_1^{\pm} + V_{\pm}) = [0, \infty)$ by theorem 3.8.1 of [14]. Then we use (26) and $\sigma_{\mathrm{ess}}(H_2) = \emptyset$ to obtain

*Proposition 4.5.* $H_{\pm}$ has an eigenvalue $E_0^{\pm} < E_\infty$ iff $H_1^{\pm} + V_{\pm}$ has a negative eigenvalue.

Results regarding the existence of bound states for a quantum tubular waveguide can now be derived using methods from one-dimensional Schrödinger operator analysis. Unfortunately, it is pointless to try to prove that $H_0$ has no bound states by demonstrating that $H_1^- + V_-$ does not have a negative eigenvalue and appealing to propositions 4.1 and 4.5. The reason for this is that such a proof would involve assuming the existence of a $\psi$ for which $\langle\psi|(H_1^- + V_-)\psi\rangle < 0$, and working to a contradiction. However, this is not feasible because in constructing a lower bound $V_-$ for the effective potential $V_{\mathrm{eff}}$, we have had to take into account attractive terms in $V_{\mathrm{eff}}$ which are not compensated for everywhere by the repulsive contribution from the more complicated torsion-dependent terms, and thus the lower bound $V_-$ will still admit bound states, as can be seen from (30).

Hence the best we can do is to demonstrate how the imposition of torsion weakens the minimax argument for the existence of curvature-induced bound states in tubular quantum waveguides which are not thin and slowly twisting. This result is applicable to the question of whether the imposition of torsion can affect the existence of bound states when the curvature does not have compact support.

*Theorem 4.6.* If

$$\frac{d}{2}[1 + \kappa_+ d]^2 \int_{-\infty}^{\infty} \max\{|\kappa'' - \kappa\tau^2|, |\kappa\tau' + 2\kappa'\tau|\}\,\mathrm{d}q_1 < \frac{[1 - \kappa_+ d]^3}{4}\int_{-\infty}^{\infty} \kappa^2(q_1)\,\mathrm{d}q_1 \tag{32}$$

then $H_0$ has a bound state.

*Proof.* From propositions 4.1 and 4.5 and the decay assumptions upon $\kappa(q_1)$ and $\tau(q_1)$, it suffices to show that $\int_{-\infty}^{\infty} V_+(q_1)\mathrm{d}q_1 < 0$. The proof follows from (29). □

In the case where $\tau = 0$, this reduces to the condition expressed by equation (4.5*b*) in [3]. Note that the left-hand side of the inequality in (32) has the potential to increase for certain conditions of $\tau$, whereas the right-hand side depends purely upon the curvature $\kappa$. This result is interesting in that it suggests, though of course it does not prove, that at least in the case of quantum waveguides for which the strict decay requirements on the curvature and torsion of Goldstone and Jaffe [15] do not hold, the torsion can act to counter the tendency of the curvature to induce bound states, a result seen before in the case of quantum strip waveguides [16]. We give an example below of a twisted quantum solenoid, where we can guarantee the existence of a bound state, subject to certain restrictions on the magnitude of the torsion.

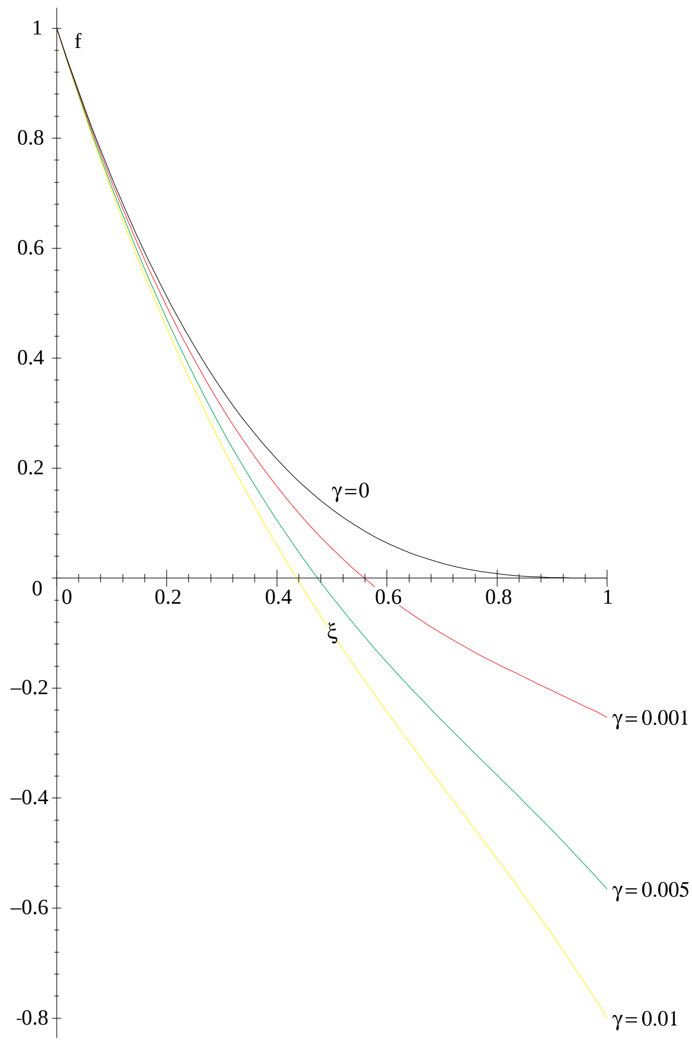


**Figure 1.** $f(\xi, \gamma)$ versus $\xi$, for small values of $\gamma$.

*Example 4.7.* Suppose we have a quantum solenoid—a tubular waveguide of radius $d$ with constant curvature $\kappa$ and torsion $\tau$ along a length $l$, connected to straight waveguides at either end. The effective potential vanishes on the straight waveguides, therefore in order to demonstrate the existence of a bound state, it suffices to show that

$$\frac{d}{2}[1+\kappa d]^2 \int_0^l \kappa\tau^2\,\mathrm{d}q_1 < \frac{[1-\kappa d]^3}{4}\int_0^l \kappa^2\,\mathrm{d}q_1. \tag{33}$$

Using the dimensionless variables $\xi$ and $\gamma$, defined by $\xi = \kappa d$ and $\tau = \gamma\kappa$, we have the following condition for the existence of a bound state:

$$f(\xi,\gamma) = (1-\xi)^3 - 2\xi\gamma^2(1+\xi)^2 > 0. \tag{34}$$

Graphing $f(\xi,\gamma)$ as a function of $\xi$ for several small values of $\gamma$, we can see that except in the case where $\kappa = 0$, condition (34) fails to be satisfied when $\gamma^2$ exceeds a certain threshold $\gamma_0^2$. Note that for curved quantum waveguides with either small or vanishing torsion and non-vanishing curvature, (34) guarantees the existence of a bound state, in agreement with [8] and [15].

Obtaining $\gamma_0$ by solving $f(\xi,\gamma_0) = 0$, we can obtain the critical torsion $\gamma_0$ for any non-vanishing value of $\xi$:

$$\gamma_0^2 = \frac{1}{2}\frac{(1-\xi)^3}{\xi(1+\xi)^2}. \tag{35}$$

The effect of increasing the torsion of such quantum solenoids while keeping the curvature constant is to require us to fabricate thinner quantum wires should we wish to guarantee a bound state in such a device, which we expect to show up as a physical resonance. Such quantum mechanical analogues of wound inductors are worthy of further investigation, in that they could be expected to have interesting physical properties.

The fact that the imposition of torsion has only a small effect in the case of thin and slowly twisting tubes, as noted by previous authors [10–12, 15], makes the question of the effect of torsion on curvature-induced bound states in a more general class of quantum waveguides a somewhat difficult one. In our introduction, we have explained why practical considerations render this question worthy of investigation. Our analysis in this work does not prove that no bound states exist if $\tau$ is sufficiently large, for a waveguide with given curvature, but we believe our preliminary result that the imposition of torsion weakens the argument for the existence of bound states opens the question for further consideration, and adds to its interest.

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